Subgoal-based Demonstration Learning for Formal Theorem Proving

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Abstract

Large language models (LLMs) present a promising pathway for advancing the domain of formal theorem proving. In this paper, we aim to improve the performance of LLMs in formal theorem proving by thoroughly examining the structure and organization of demonstrative in-context examples. We introduce a subgoal-based demonstration learning framework, specifically designed to enhance the efficiency of proof search in LLMs. First, drawing upon the insights of subgoal learning from reinforcement learning and robotics, we propose the construction of distinct subgoals for each demonstration example and refine these subgoals in accordance with the pertinent theories of subgoal learning. Second, we build upon recent advances in diffusion models to predict the optimal organization, simultaneously addressing two intricate issues that persist within the domain of demonstration organization: subset selection and order determination. Our integration of subgoal-based learning has notably increased proof accuracy from 38.9% to 44.1% on the miniF2F benchmark. Furthermore, the adoption of diffusion models for demonstration organization can lead to an additional enhancement in accuracy to 45.5%, or a 5× improvement in sampling efficiency compared to previously established methods.

1. Introduction

With the ongoing investigation into the complex reasoning abilities of large language models (LLMs), there has been a surge of interest in employing them for mathematical problem solving. This has led to an expanding body of research that underscores the remarkable potential of LLMs in this domain (Lewkowycz et al., 2022; Wu et al., 2022a; Jiang et al., 2023).

In this context, we pay particular attention to the problem of formal theorem proving (Polu & Sutskever, 2020). Not only due to its extensive applications in various scientific domains, such as software verification (Klein et al., 2009) and research-level mathematics (Castelvecchi et al., 2021), but also stems from its advantage of leveraging interactive proof assistants (Paulson, 2000) to automatically validate proofs generated by models, thereby eliminating the need for human proof checking. This significantly facilitates the generation of machine-verifiable proofs via LLMs.

The complexity of automated theorem proving comes from the necessity of searching through a vast space of possible logical statements and proof methods, to determine the truth-value of a given theorem. LLMs reduce the difficulty of the searching problem by factorizing the formal proof automation task into two in-context learning (§5.2) problems (Wu et al., 2022a; Jiang et al., 2023; First et al., 2023). Given a mathematical statement, an LLM first generates its informal proof as a draft. It then generates a formal sketch based on this draft, which is ready for an off-the-shelf prover to verify its correctness automatically.¹ In both of these steps, the quality of the demonstrative in-context examples either written by humans or generated by machines is the key to the performance of the system.

In this paper, we seek to improve the efficacy of LLMs in formal theorem proving by delving deeper into the format and the organization of these demonstrative in-context examples. We present a subgoal-based demonstration learning framework, comprising two main components. First, we restructure an informal proof into a subgoal-based proof (Figure 1(a)), drawing upon the insights of subgoal learning from reinforcement learning and robotics, where studies show that breaking down complex tasks into smaller yet more uniformed subgoals enhances the learning efficiency of the agents (Eysenbach et al., 2019; Zhang et al., 2021). To construct subgoal-based proofs that can be easily processed and handled by LLMs, we start with human-written informal proofs and then iteratively refine them through interaction with ChatGPT (OpenAI, 2022), guided by the subgoal learning theory (§2.1). Second, a recent study (Wu et al., 2022b) points out that the selection and the ordering of

¹In practice, the informal proof often serves as inline comments in the formal sketch to better guide the generation procedure.
Given a theorem statement $x$, the goal of proof synthesis is to generate a formal sketch $y$ which can be verified by an off-the-shelf automated theorem prover (e.g., Sledgehammer) (Jiang et al., 2023). In this section, we elaborate on the subgoal-based demonstration learning framework that consists of two key components, subgoal-based proof (§2.1) and demonstration reorganization (§2.2). The subgoal-based proof replaces the informal proof, breaking down a complex problem into smaller subgoals that offer more fine-grained and uniform guidance to the LLMs. The demonstration reorganization takes place in the stage of generating the formal sketch based on the subgoal-based proof. This procedure is non-trivial. Given the limited context length of the LLMs, selecting relevant yet diverse demonstration examples has a significant impact on the final pass rate of these formal sketches. We denote the set of all $N$ demonstration examples by $E = \{E_1, E_2, \cdots, E_N\}$. Each of them contains a mathematical statement, an informal proof (or a subgoal-based proof), and a formal sketch. In the remainder of this section, we first describe the iterative refinement process that produces the subgoal-based proofs given the informal proof, guided by the principles in the subgoal learning theory (Zhang et al., 2021). We then explain our solution to the demonstration reorganization. Starting from collecting arrangements that have yielded successful proofs, we use these as training data for a diffusion model, which progressively determines the most favorable reorganization during inference.

### 2.1. Subgoal-based Proof

The significance of LLMs to formal theorem proving is that they grant us the ability to leverage informal proofs to guide formal theorem proving, which otherwise has to be based on expensive heuristics-based brute-force search. Despite considerable progress (Lewkowycz et al., 2022; OpenAI, 2023), this approach suffers from the flawed informal proofs generated by the LLMs (Jiang et al., 2023). We propose to use subgoal-based proofs to replace the informal proofs, where the subgoals are strictly aligned with the states in the automatic provers. Following Zhang et al. (2021), we seek to obtain a valid sequence of subgoals which satisfies the condition that each subgoal in this sequence should be reachable from the initial state (i.e., the statement) and attain the final state (i.e., the passing state of the proof). These valid
sequences integrate the guidance from the LLMs better with the search space of the automatic theorem provers, thereby leveraging the ability of the LLMs to the maximum extent. However, it is non-trivial to get these valid subgoal proofs as human-written subgoals often fall short of the above constraints. To address this problem, we iteratively refine the subgoal proof, in the spirit of self-play in reinforcement learning (Silver et al., 2016), making calls to both the LLM and the off-the-shelf automated theorem prover.

Subgoal Refinement. We start with manually written subgoal-based proofs, and denote these as the initial seed set \(\{E_i^{(0)}\}_{i=1}^N\). This set contains subgoal-based proofs formed on the informal proofs and the statement, yet not guaranteed to be a valid sequence. We denote the sequence of subgoals in an instance as \((s_0, s_1, s_2, \ldots, s_\Delta, s_{\Delta+1})\), where \(\Delta\) is the total number of subgoals and \(s_0\) and \(s_{\Delta+1}\) are two special subgoals that align with the initial and final states of the automatic prover. During the \(k\)-th iteration, we randomly select a subset of instances from the previous iteration \(\{E_i^{(k-1)}\}_{i=1}^N\) as the in-context demonstration for the LLM to generate subgoals for a given instance. According to the definition, \(s_i\) is considered to be a valid subgoal if and only if it can be reached from \(s_0\) and can reach \(s_{\Delta+1}\). Therefore, for each of the subgoals, we recursively call the proof assistant to verify the validity of the most recently developed subgoal and only after \(\Delta\) recursions we can obtain the new valid sequence of subgoals and add that into the next iteration as \(E_i^{(k)}\). This process improves the consistency of the derived subgoals in style, thus making it easier for the LLM to learn from in the inference stage. We provide a detailed description of the algorithm in the Appendix.

2.2. Demonstration Reorganization

The demonstration examples can be lengthy in formal theorem proving. If we assume a maximum context length of 3072 tokens, only 4.79 examples on average can be included. Our experiments echo the findings by Wu et al. (2022b). These instance-based demonstration examples have a significant impact on performance. Only certain orders of carefully chosen demonstration examples lead to successful theorem proving. Consequently, identifying the optimal subset from the pool and ordering them into meaningful in-context demonstration examples is of great significance, which unfortunately is an NP-complete problem. We formulate the demonstration reorganization problem as finding the (Sub)hamiltonian graph where the nodes represent demonstration examples, and a traversal following the path corresponds to the selection and ordering of them. Building upon the recent success of applying diffusion models in addressing NP-complete problems (Graikos et al., 2022; Sun & Yang, 2023), we further treat this problem as a diffusion process on the graph. This solution has two main advantages. First, it addresses the example selection and ordering problem simultaneously. Second, the inference can be performed in parallel, which greatly reduces the time of discovering the optimal arrangement given the demonstration examples. We start by collecting successful pairs of in-context demonstration example organization and the corresponding statement \(x\) as the training data for the diffusion model. We randomly organize (select and order) the demonstration examples and query the LLM to see if it can generate the proof successfully. The passing cases will be used as the starting configuration \(\psi_0\) in the diffusion process given the statement \(x\).

Training. The aim of employing diffusion models is to predict the optimal organization, denoted as \(\psi_0^3\) conditioning on the theorem statement \(x\). From the standpoint of variance inference, diffusion models adopt the following formulations to model \(p_\theta(\psi|x)\),

\[
p_\theta(\psi_0|x) := \int p_\theta(\psi_{0:T}|x)d\psi_{1:T},
\]

where \(\psi_1, \ldots, \psi_T\) serve as latent variables with the same dimensionality as \(\psi_T\). The learned reverse process progressively denoises these latent variables to reconstruct \(\psi_0\). This procedure can be formalized as follows,

\[
p_\theta(\psi_{0:T}|x) = p(\psi_T) \prod_{t=1}^T p_\theta(\psi_{t-1}|\psi_t, x).
\]

The forward process gradually corrupts \(\psi_0\) to generate noised latent variables,

\[
q(\psi_{1:T}|\psi_0) = \prod_{t=1}^T q(\psi_1|\psi_{t-1}).
\]

The goal of the training process is to maximize the evidence lower bound (ELBO),

\[
\mathbb{E}[\log p_\theta(\psi_0|x)] \geq \mathbb{E}_q \left[ \log \frac{p_\theta(\psi_{0:T}|x)}{q(\psi_{0:T}|\psi_0, x)} \right] = \mathbb{E}_q \left[ \log p_\theta(\psi_{0:T}|x) - \sum_{t=1}^T D_{KL}(q(\psi_{t-1}|\psi_t, \psi_0)\|p_\theta(\psi_{t-1}|\psi_t, x)) \right] .
\]

We employ a Graph Neural Network (GNN) for the encoding and denoising process of the graph. Following Austin et al. (2021), we adopt discrete diffusion models to model binary random variables.

Inference. During the inference stage, we obtain samples \(\psi \sim p_\theta(\psi_0|x)\) and subsequently reconstruct the order of demonstration examples from \(\psi\). We then incorporate \(\psi_0\) represents the sequence of demonstrations that, when fed into the prompt, results in a successful formal sketch.
examples sequentially into the LLM context, and define the output of the demonstration organization module as the sequence of examples upon reaching the LLM length constraint. More details of the implementation of the diffusion model, the implementation of GNN, and techniques used in the sampling process of $\psi$ can be found in the Appendix.

3. Experiments

3.1. Formal Environment

Interactive Theorem Provers. Interactive Theorem Provers (ITPs), such as Isabelle (Paulson, 1994), constitute the backbone of contemporary mathematical verification systems. They facilitate the integration of mathematical definitions and theorems into a consistent logical framework, such as Higher-Order Logic or Dependent Type Theory, which is operationalized by their kernels. The kernel plays a pivotal role in the verification process, meticulously examining each theorem to ascertain its recognition by the ITP and thereby ensuring the integrity of the system. The theorem-proving process within an ITP is characterized by the articulation of the theorem in the ITP’s programming language, followed by an iterative simplification into more manageable objectives or subgoals. The theorem is deemed proven once it can be distilled down to pre-established facts. The selection of Isabelle for our paper is motivated by its intuitive interface, its compatibility with a range of logical frameworks, and its comprehensive library of formalized mathematics.

Sledgehammer. Sledgehammer (Paulsson & Blanchette, 2012) serves as a powerful tool for automating reasoning within the interactive theorem prover Isabelle. It functions by transmuting the goals encapsulated in Isabelle/HOL’s higher-order logic into alternative logic, such as first-order logic. These transmuted goals are then passed to off-the-shelf automated theorem provers, including E, CVC4, Z3, Vampire, and SPASS. If any of these automated theorem provers successfully derive the proof in their respective formats, Sledgehammer undertakes the task of reconstructing the proof within the Isabelle/HOL framework using certified provers, namely meson, and smt. This reconstructed proof, being more interpretable to humans, significantly enhances the system’s usability, thereby contributing to the efficiency and effectiveness of (interactive) theorem proving.

3.2. Dataset and Evaluation

Dataset. We evaluate our approach using the miniF2F dataset (Zheng et al., 2021), which comprises 488 formal mathematical problems derived from high-school competitions, expressed in three formal languages: Lean, HOL-Light, and Isabelle. The dataset is divided into a validation and a test set, each including 244 problems. The problems within the dataset are sourced from three distinct categories: 260 problems are extracted from the MATH dataset (Hendrycks et al., 2021), 160 problems are extracted from actual high-school mathematical competitions (AMC, AIME, and IMO), and 68 problems are crafted to mirror the difficulty level of the aforementioned competitions.

Evaluation. The task at hand entails the generation of formal sketches for problems in the miniF2F dataset. The validity of a formal sketch depends on two criteria: first, the absence of “cheating” keywords such as “sorry” and “oops” that prematurely terminate a proof before its completion; second, the capacity of the interactive theorem prover Isabelle to authenticate the corresponding formal statement with the proof. To make working with Isabelle easier, we use the Portal-to-Isabelle API, as introduced by Jiang et al. (2023). Given the absence of a training split in the miniF2F dataset, we leverage optimal organizations that yield successful proofs from the miniF2F-valid set to train the diffusion model. As proposed by Lample et al. (2022), we employ the cumulative pass rate as a measure for the results obtained from performing inference using diffusion models on the miniF2F-valid set. This involves integrating the pass rates from both the data collection stage for training and the inference stage. When it comes to other scenarios, namely conducting inference on the miniF2F-test or cases where the diffusion model is not employed, we simply provide the pass rate.

3.3. Baselines

We leverage the following baselines to substantiate the effectiveness of our proposed methodology:

Symbolic Automated Provers. We first employ Sledgehammer, a proof automation tool that is extensively utilized within the Isabelle environment. We adhere to the default configuration of Sledgehammer as provided in Isabelle2021, which encompasses a 120-second timeout and a suite of five automated theorem provers (Z3, CVC4, SPASS, Vampire, E). In alignment with Jiang et al. (2023), we employ Sledgehammer supplemented with heuristics, integrating 11 prevalent tactics (i.e., auto, simp, blast, fastforce, force, eval, presburger, sos, arith, linarith, auto simp: field simps) with Sledgehammer. If all the tactics fail or take longer than 10 seconds, the system reverts to the base Sledgehammer.

Search-based Methods. In addition to the above, we incorporate baselines that utilize Monte-Carlo tree search (Silver et al., 2016) to discover the proof. This includes Thor (Jiang et al., 2022) and another version of Thor that employs an expert iteration on autoformalized data (i.e., Thor+expert iteration (Wu et al., 2022a)). Thor combines
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Table 1. Pass rates on the miniF2F dataset with Isabelle. Numbers in bold denote the best performance. Numbers with a * correspond to the cumulative pass rate (Lample et al., 2022) since the evaluated statements are part of the training for diffusion models. See §3.2 for more details about cumulative pass rate. † denotes the concurrent work at the time of submission.

<table>
<thead>
<tr>
<th>Method</th>
<th>valid</th>
<th>test</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sledgehammer</td>
<td>9.9%</td>
<td>10.4%</td>
</tr>
<tr>
<td>Sledgehammer+heuristic</td>
<td>18.0%</td>
<td>20.9%</td>
</tr>
<tr>
<td>Thor</td>
<td>28.3%</td>
<td>29.9%</td>
</tr>
<tr>
<td>Thor + expert iteration</td>
<td>37.3%</td>
<td>35.2%</td>
</tr>
<tr>
<td>DSP (540B Minerva)</td>
<td>42.6%</td>
<td>38.9%</td>
</tr>
<tr>
<td>LEGO-Prover</td>
<td>52.4%</td>
<td>45.5%</td>
</tr>
<tr>
<td>Lyra†</td>
<td>52.8%</td>
<td>44.2%</td>
</tr>
<tr>
<td>Ours</td>
<td>48.0%</td>
<td>45.5%</td>
</tr>
</tbody>
</table>

Table 2. Ablation results on the miniF2F dataset with Isabelle. Numbers with a * correspond to the cumulative pass rate.

<table>
<thead>
<tr>
<th>Ablation</th>
<th>valid</th>
<th>test</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ours</td>
<td>48.0%* (±0.4)</td>
<td>45.5% (±0.6)</td>
</tr>
<tr>
<td>- subgoal &amp; diffusion</td>
<td>41.4% (±0.9)</td>
<td>38.7% (±1.2)</td>
</tr>
<tr>
<td>- subgoal</td>
<td>44.3%* (±0.7)</td>
<td>40.6% (±0.6)</td>
</tr>
<tr>
<td>- diffusion</td>
<td>46.9% (±1.3)</td>
<td>44.1% (±0.9)</td>
</tr>
</tbody>
</table>

language models with automatic theorem provers to overcome the challenge of selecting beneficial premises from a vast library. Thor+expert iteration enhances a neural theorem prover by training it on theorems that have been automatically formalized.

LLM-based Method. Lastly, we incorporate three LLM-based baselines: Draft, Sketch, and Prove (DSP) (Jiang et al., 2023), LEGO-Prover (Xin et al., 2023), and Lyra (Zheng et al., 2023). DSP utilizes the 540B Minerva model (Lewkowycz et al., 2022) to generate informal proofs, which are then transformed into formal sketches. LEGO-Prover focuses on incrementally creating reusable theorems, streamlining the theorem proving process through the utilization of previously established results. Lyra improves the verification process by integrating error messages from external verifiers, facilitating proof optimization. Notably, all three methods, similar to our approach, employ the Sledgehammer tool, maintaining consistency in tool usage across the methodologies. For a fair comparison, we use versions of these baselines configured with 100 autoformalization attempts and without relying on ground-truth informal proofs.

We exclude methods such as HyperTree Proof Search (HTPS) (Lample et al., 2022) and GPT-I with expert iteration (Polu et al., 2022), which are implemented using Lean (de Moura et al., 2015), a different interactive theorem prover. The disparity in tactics and automation between Lean and Isabelle renders them not directly comparable to our method.

3.4. Implementation Details

Throughout our work, we employ ChatGPT (i.e., the gpt-3.5-turbo-0301 version) as the LLM. For the creation of the formal sketch, the temperature and max_tokens parameters of gpt-3.5-turbo-0301 are set to 0 and 1024, respectively. For each subgoal proof, we make one formal sketch attempt, as suggested by previous literature (Jiang et al., 2023). In terms of the establishment of the subgoal-based proof, we set the number of refinement iterations to be 15, with the number of demonstration examples, denoted as N, being set to 61. For demonstration organization, we employ a randomized demonstration organization approach to generate proofs for 116 distinct statements on miniF2F-valid, which yield 137 successful proofs. We then partition the corresponding demonstration contexts into a training set and a validation set, comprising 81 and 56 instances respectively.

The training of our diffusion models is conducted with a learning rate of 5e-4, a batch size of 16, and throughout 50 epochs. We set the number of diffusion steps, represented as T, to 80. We employ an early stopping strategy on the validation set and report the performance averaged in three repetitive experiments.

3.5. Main Results

The experiment results, as shown in Table 1, yield several key observations: (1) Our method outperforms DSP,
with pass rates of 48.0% on miniF2F-valid and 45.5% on miniF2F-test. This superior performance is attributable to our novel application of diffusion models for demonstration reorganization coupled with subgoal-based proof; (2) The methods Thor and Thor + expert iteration struggle due to the enormously large action space. This space significantly overshadows that of games, thereby posing challenges to the comprehensive utilization of the Monte Carlo tree search. Consequently, these methods underperform when compared to LLM-based methods; (3) DSP has pioneered the introduction of the informal proof, a critical step in the LLM-based formal theorem proving task. However, human-written informal proofs do not offer optimal compatibility with large language models. Our method, grounded in the subgoal-learning theory, is capable of inferring subgoal-based proofs that are more amenable to large language models. Our method, grounded in the subgoal-learning theory, is capable of inferring subgoal-based proofs that are more amenable to large language models; and (4) LEGO-Prover and Lyra, which achieve on-par performance with our model, are concurrent work with complementary focuses. LEGO-Prover focuses on the incremental creation of reusable theorems, and Lyra enhances verification via error feedback integration. The orthogonal nature presents a promising direction for integrating our approach with theirs in the future.

4. Analysis

4.1. Ablation Study

In our ablation study, we examine four variations of our model on the miniF2F dataset, as detailed in Table 2. The models include our full method (Ours), and three variants: one excluding the subgoal-based proof, denoted as “-subgoal”, which utilizes informal proofs instead; one without the diffusion method, referred to as “-diffusion”, employing a randomized selection for demonstrations; and one, denoted as “-subgoal & diffusion”, representing the basic form of gpt-3.5-turbo, where both the subgoal-based proof and diffusion method are omitted. To quantify the uncertainty associated with our method, we have supplemented our results with standard deviation values. For the variants employing diffusion models (“Ours” and “-subgoal”), we trained three separate diffusion models as detailed in §3.4. For the variants without diffusion models (“-subgoal & diffusion” and “-diffusion”), the experiment was repeated three times.

Our full model achieves the highest performance on both validation and test sets. This underscores the importance of both subgoal-based proof and demonstration reorganization. The model without both components showed the lowest performance, further emphasizing the significance of these components. The models missing either the subgoal-based proof or reorganization components also show decreased performance, indicating the substantial role of each component. Additionally, it is noteworthy that the methods incorporating diffusion models exhibit relatively more stable performance, affirming the stability benefits of these models in our study.

4.2. On the Effect of Subgoal-based Proof

We further use four different variants to explore the impact of subgoal-based proof. Figure 2(a) displays the results of this experiment, where “informal” denotes the utilization of informal proofs instead of subgoal-based proof, and “diff” indicates the integration of demonstration reorganization.
Statement: Suppose \( n \) is a natural number such that \( n \) leaves a remainder of 3 when divided by 7. Show that \((2n+1)\) leaves a remainder of 0 when divided by 7.

Subgoal-based Proof:
Step 1: Express \( n \) as \( 7k + 3 \) for some \( k \in \mathbb{N} \)
Step 2: Substitute the expression for \( n \) into the expression for \( 2n + 1 \) and simplify.
Step 3: Show that the resulting expression is divisible by 7.

Formal Sketch:
\[
\begin{align*}
\text{fixes} & \quad n :: \text{nat} \\
\text{assumes} & \quad h_0 : \quad \text{"}n \mod 7 = (3::\text{nat})\text{"} \\
\text{shows} & \quad \text{"}(2^n+1) \mod 7 = (0::\text{nat})\text{"} \\
\text{proof -} \\
\text{(* Step 1: Express } n \text{ as } 7k + 3 \text{ for some } k \in \mathbb{N}. *) \\
\text{have } \text{"}n = 7 \ast (n \text{ div } 7) + (n \text{ mod } 7)\text{"} \\
\text{using} & \quad \text{div_mult_mod_eq}[\text{of } \text{"}n \text{"}, \text{"}7\text{"}] \text{ sledgehammer} \\
\text{also have } & \quad \text{"}... = 7 \ast (n \text{ div } 7) + 3\text{"} \\
\text{using} & \quad h_0 \text{ sledgehammer} \\
\text{finally have } & \quad \text{c0: } \text{"}n = 7 \ast (n \text{ div } 7) + 3\text{"}. \\
\text{(* Step 2: Substitute the expression for } n \text{ into the expression for } 2n + 1 \text{ and simplify. *)} \\
\text{have } & \quad \text{"}2^n+1 = 2^n(7\ast(n \text{ div } 7) + 3) + 1\text{"} \\
\text{using} & \quad \text{c0 sledgehammer} \\
\text{also have } & \quad \text{"}... = 34^n(\text{c0 div } 7) + 7\text{"} \text{ sledgehammer} \\
\text{also have } & \quad \text{"}... = 7^n(\text{c0 div } 7) + 1\text{"} \text{ sledgehammer} \\
\text{finally have } & \quad \text{c1: } \text{"}2^n+1 = 7^n(2^n(\text{c0 div } 7) + 1)\text{"}. \\
\text{(* Step 3: Show that the resulting expression is divisible by 7. *)} \\
\text{show } & \quad \text{"}(2^n+1) \mod 7 = 0\text{"} \\
\text{using} & \quad \text{c1 sledgehammer} \\
\end{align*}
\]
\[
\text{qed}
\]

Figure 3. A formal sketch generated by our proposed method.

The results indicate a significant difference between the approaches that incorporate subgoal-based proof (“subgoal” and “subgoal+diff”) and those that do not (“informal” and “informal+diff”). This trend remains consistent across all LLM call numbers, suggesting a noteworthy positive effect of subgoal-based proof on the overall performance of our method.

4.3. On the Effect of Demonstration Reorganization

To further investigate the effect of a diffusion model for demonstration reorganization, we draw a comparative analysis between its performance and two alternative in-context learning methodologies: the Graph Neural Network (GNN) and the Top-K. The Top-K method calculates Ada embeddings for test problem statements and in-context examples, subsequently selecting the top-k most similar examples, as measured by dot product, ensuring those with higher similarity are positioned nearer to the test problem in the prompt. The GNN is congruent with a modified version of our proposed model when the inference diffusion step is set to 1, while the efficacy of the Top-K methodology has been extensively substantiated in the literature (Liu et al., 2021). Figure 2(b) presents the empirical results, manifesting that the diffusion model’s performance increment diminishes as the number of LLM calls escalates to 100. This phenomenon stems from the fact that the module is trained on data collated from successful proofs via randomized organization sampling. Consequently, it may encounter difficulties in discerning the optimal organization for data that deviates significantly from its training dataset. Nevertheless, this limitation does not overshadow the potential of diffusion models to economize the number of LLM calls. Notably, with demonstration reorganization, our method exhibits an impressive capability of successfully deriving proofs for 94 problems (equivalently, a pass rate of 38.5%), with a mere 20 LLM calls. Remarkably, this result is comparable with that of the DSP method, which necessitates \(5 \times\) the number of LLM calls.

4.4. Case Study

To better comprehend the efficacy of our proposed method, we present a formal sketch of a problem that remains un-
proven by earlier state-of-the-art methods. As demonstrated in Figure 3, it is apparent that our strategy successfully decomposes the complex objective into three manageable subgoals, each solvable by the LLM. We provide additional comprehensive examples in the Appendix.

5. Related Work

5.1. Machine Learning for Formal Theorem Proving

Formal theorem proving has seen significant advancements through machine learning, with efforts focusing on enhancing proof search strategies (Polu & Sutskever, 2020; Polu et al., 2022; Jiang et al., 2022) and employing Large Language Models (LLMs) for autoformalization (Jiang et al., 2023). Proof search improvements are marked by the development of self-supervised strategies in Expert Iteration (Polu et al., 2022) and PACT (Han et al., 2021), alongside integrations of language models with automated provers in HyperTree Proof Search (HTPS) (Lample et al., 2022) and Thor (Jiang et al., 2022), complemented by transformer-based premise selection in Magnushammer (Mikula et al., 2023). However, scalability remains a challenge due to the increasing complexity of theorems. In parallel, the application of LLMs for autoformalization and proof generation has been explored, with Wu et al. (2022a) and Jiang et al. (2023) demonstrating the conversion of mathematical problems into formal specifications. Baldur (First et al., 2023) goes further by producing full proofs and enhancing proving capabilities with a proof repair model. Additionally, LEGO-Prover (Xin et al., 2023) and Lyra (Zheng et al., 2023) offer unique contributions in theorem proving, focusing on the incremental development of reusable theorems and the integration of error messages from external verifiers for proof post-processing, respectively. Nevertheless, these approaches have yet fully optimized the format and the organization of demonstration examples when invoking LLMs. Our work aims to address this gap by introducing a subgoal-based demonstration learning framework that refines the use of LLMs in formal theorem proving.

5.2. In-context Learning

In-context Learning (ICL) primarily investigates two aspects: the selection and arrangement of in-context examples. For example selection, Liu et al. (2021) introduce a retrieval-based method for prompt selection, enhancing the semantic relevance over random selection. This concept was expanded by Rubin et al. (2021) who utilize a pre-trained language model for effective prompt retrieval. Additionally, Sorensen et al. (2022) propose a novel template selection method based on maximizing mutual information, circumventing the need for labeled examples or direct model access. Su et al. (2022) present a two-step framework focusing on efficiency, selecting examples from unlabeled data and retrieving task-specific examples during testing. Furthermore, Agrawal et al. (2022) develop strategies for machine translation, emphasizing the importance of example quality and domain relevance while highlighting the detrimental effects of irrelevant examples. Concerning example arrangement, Zhao et al. (2021) address the instability in few-shot learning results caused by example order, proposing a calibration method. Extending this, Lu et al. (2021) explore the sensitivity of prompt order in few-shot learning contexts. Even though previous efforts have made remarkable progress in either choosing or sequencing in-context examples, our research sets a new precedent by combining both elements. In this paper, we step out of these isolated areas of concentration, looking into an approach based on diffusion models that effectively tackles both the challenges of selection and ordering at the same time.

5.3. Subgoal Learning

Subgoal learning significantly enhances AI systems’ ability to address complex tasks by introducing efficiency and structure in reinforcement learning (RL). Theoretical contributions highlight the computational benefits of subgoal rewards (Zhai et al., 2022), optimal structuring for hierarchical RL (Wen et al., 2020), option selection complexities (Jinnai et al., 2019a), and temporal abstraction integration (Fruit et al., 2017). Empirical efforts focus on subgoal exploration, planning, and curriculum learning, leveraging strategies for optimal discovery and decision-making through coverage minimization (Jinnai et al., 2019b), dynamical distance learning (Hartikainen et al., 2019), entropy maximization (Pitis et al., 2020), asymmetric self-play (OpenAI et al., 2021), and innovative planning algorithms like SoRB (Eysenbach et al., 2019), DC-MCTS (Parascandolo et al., 2020), PAIR (Li et al., 2022), and goal-oriented MCTS with hindsight experience replay (Mor0 et al., 2022). Curriculum learning research aims at progressively enhancing subgoal complexity to optimize learning pathways, with techniques for automatic curriculum generation and complexity scaling (Zhang et al., 2020; 2021). While there have been preliminary efforts to apply similar principles in the construction of prompts for LLMs (Khot et al., 2022), the deployment of subgoal learning theories to manage intricate tasks, such as formal theorem proving, remains largely unexplored. Our work pioneers the use of subgoal learning in this domain, with a focus on format and organization.

6. Conclusion

In this paper, we have developed a subgoal-based demonstration learning framework that significantly enhances LLMs’ efficacy in formal theorem proving. Our approach combines insights from subgoal learning and diffusion models, effectively addressing the challenges of demonstration formatting.
and organization. As a result, we achieve a 17.0% relative improvement in proof pass rate on the miniF2F benchmark and a 5× improvement in sampling efficiency.

Acknowledgements

We would like to thank the HKU NLP group and the anonymous reviewers for their valuable suggestions that greatly helped improve this work. We are also grateful to Wei Bi for generously providing the CPU servers that significantly contributed to the computational resources required for this research. This work is partially supported by the joint research scheme of the National Natural Science Foundation of China (NSFC) and the Research Grants Council (RGC) under grant number N_HKU714/21.

Impact Statement

This work presents a novel algorithm for automated formal proof generation. Its impacts potentially span across mathematical research and software engineering: This algorithm may assist mathematicians in writing rigorous mechanized proofs, complementing their conceptual exploration; While our work is primarily focused on mathematics at present, there is potential for its application to extend to formal verification of software systems. Future research can focus on enhancing the algorithm’s accuracy, broadening its application to more complex theorems, and exploring synergies with other AI techniques.

References


A. More Details about Subgoal-based Proof

We provide a detailed description of the subgoal refinement method (see §2.1) through Algorithm 1. In the $k$-th iteration, we construct demonstration examples $\{E_i^{(k)}\}_{i=1}^N$ using improved subgoal-based proofs. To construct $E_i^{(k)}$, we first extract the statement and formal sketch from $E_i^{(k-1)}$, then use an LLM to generate subgoals. Afterward, a Refine module is called to confirm the validity of the created subgoals and adjust any subgoals identified as infeasible.

A subgoal $s_\Delta$ is deemed “reachable” from $s_1$ if, for any $s_i$ where $1 < i \leq \Delta$, the ATP solvers can bridge the gap between $s_{i-1}$ and $s_i$, under the assumption that all preceding subgoals $s_j$ (where $j < i$) are true. This concept is crucial in cases where the proof structure necessitates the independent verification of lemma a, b from $s_1$, followed by the substantiation of $s_2$ utilizing both lemma a and b. In such scenarios, as per our definition of reachability, the verification of $(s_{i-1}, s_i)$ inherently presumes the truth of all prior subgoals, denoted as all $s_j$ with $j < i$, which are encapsulated within the context $C$ in VERIF_AND_CORRECT. Specifically, the verification of $(s_a, s_b)$ presupposes $s_1$ and $s_a$, and the verification of $(s_b, s_2)$ presupposes $s_1$, $s_a$, and $s_b$. If the ITP validates the steps, the subgoal is deemed reachable; otherwise, it is considered unattainable.

In practice, the reachability of subgoals is tested using the VERIFY function in Algorithm 4, which integrates both an Interactive Theorem Prover (ITP) and a Large Language Model (LLM). It amends the formal sketch for a subgoal, assigns the LLM to complete the missing segments, and then validates these with the ITP. If the ITP affirms the steps, the subgoal is deemed reachable; otherwise, it is considered unattainable. In instances where the LLM consistently fails to generate provable subgoals for the theorem prover, the VERIFY_AND_CORRECT function will revert to returning the original subgoal pair $(s_1, s_{i+1})$.

We present an example to elucidate this process further (see Figures 4 to 12). As shown in Figure 4, the LLM creates two subgoals for the theorem amc12a_2003_p4, leading to $(s_0, s_1, s_2, s_3)$. Refining these subgoals involves calling VERIFY_AND_CORRECT($s_0, s_1$) to improve the subgoal $s_1$. This is depicted in Figures 5 to 12. We first use the LLM to reconstruct the subgoal related to the first step, but this attempt fails (Figure 5). Then we break down the subgoal $s_1$ into three more detailed subgoals (Figure 6), each of which is then verified using the same LLM (Figures 7 to 9). Due to the unsuccessful reconstruction of the second subgoal (Figure 8), it is further broken down into two more specific subgoals (Figure 10). The last two subgoals pass the verification process successfully (Figures 11 and 12). Finally, the output of VERIFY_AND_CORRECT($s_0, s_1$), namely $S_0^{0-1}$, is defined as the set that includes the steps from 1 to 4 shown in Figure 12.

Algorithm 1 Iterative Subgoal Refinement

<table>
<thead>
<tr>
<th>Requires:</th>
<th>Extract</th>
<th>extraction of statement and formal sketch</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Compose</td>
<td>composing of a statement, formal sketch</td>
</tr>
<tr>
<td></td>
<td>INITIALIZE_SUBGOALS</td>
<td>generate subgoals with a LLM</td>
</tr>
</tbody>
</table>

function ITERATIVE_REFINEMENT($\{E_1^{(0)}, E_2^{(0)}, \ldots, E_N^{(0)}\}$)

for $k$ in 1, 2, ..., $K$ do

for $i$ in 1, 2, ..., $N$ do

$x, y \leftarrow$ EXTRACT($E_i^{(k-1)}$)

$s_0, s_1, \ldots, s_\Delta, s_{\Delta+1} \leftarrow$ INITIALIZE_SUBGOALS($x, y, E_i^{(k-1)}$)

$S^{0-(\Delta+1)} \leftarrow$ REFINE($\{s_0, s_{\Delta+1}, \{s_1, s_2, \ldots, s_\Delta\}\}$)

$E_i^{(k)} \leftarrow$ COMPOSE($x, y, S^{0-(\Delta+1)}$)

end for

end for

return $\{E_1^{(K)}, E_2^{(K)}, \ldots, E_N^{(K)}\}$

end function

---

To simplify the illustration, we leave out redundant demonstration examples.

---

1To simplify the illustration, we leave out redundant demonstration examples.
Algorithm 2 Refinement Algorithm

**Requires:** 
- VERIFY_AND_CORRECT: verify the validness of the subgoals and correct them if necessary.

**function** 
\textsc{Refine}(s_i, s_{i+1}, \{s_{i+1}, \ldots, s_j\})

\begin{itemize}
  \item if \(i = j\) then return VERIFY_AND_CORRECT(s_i, s_{i+1})
  \item \(S^{i \rightarrow i+1} \leftarrow \text{Refine}(s_i, s_{i+1}, \{\})\)
  \item \(S^{i+1 \rightarrow j+1} \leftarrow \text{Refine}(s_{i+1}, s_{j+1}, \{s_{i+2}, \ldots, s_j\})\)
\end{itemize}

return \(S^{i \rightarrow i+1} \cup S^{i+1 \rightarrow j+1}\)

end function

Algorithm 3 Verify and Correct

**Requires:** 
- VERIFY verify if \(s_{i+1}\) is reachable from \(s_i\)
- CORRECT correct the pair of subgoals if necessary
- \(M\) the maximum number of LLM calls, which is set to 10 in our experiments
- \(C\) context including formal sketch and current subgoal-based proof

**function** VERIFY_AND_CORRECT(s_i, s_{i+1}, C)

\begin{itemize}
  \item budget \(\leftarrow M\)
  \item \(Q \leftarrow \emptyset\) → A priority queue; lower pos values indicate proximity to the formal statement
  \item \(Q.\text{PUSH}((-C.\text{pos}(s_{i+1}), (s_i, s_{i+1})))\) → Push subgoals based on their distance to the formal statement
  \item valid_subgoals \(\leftarrow \emptyset\)
  \item while \(Q \neq \emptyset\) and budget > 0 do
    \item \(s, s' \leftarrow Q.\text{POP}()\) → Retrieve subgoal closest to the formal statement
    \item budget \(\leftarrow\) budget – 1 → LLM call consumed by Verify
    \item if VERIFY(s, s', C) then
      \item valid_subgoals.APPEND((s, s'))
    \item else
      \item subgoals \(\leftarrow \emptyset\)
      \item subgoals, C \(\leftarrow\) CORRECT(s, s', C) → Fix subgoal \(s'\) or generate granular subgoals, then update the context
      \item for \((s_j, s_{j+1})\) in subgoals do
        \item \(Q.\text{PUSH}((-C.\text{pos}(s_{j+1}), (s_j, s_{j+1})))\)
      \item end for
    \item end if
  \item end while
  \item if valid_subgoals \(\neq \emptyset\) then
    \item return valid_subgoals
  \item else
    \item return [(s_i, s_{i+1})]
  \item end if
\end{itemize}

end function

Algorithm 4 Verify

**Requires:** 
- ITP Interactive Theorem Prover, i.e., Isabelle in our experiments
- LLM Large Language Model, i.e., GPT-3.5-turbo in our experiments

**function** VERIFY(s_i, s_{i+1}, C)

\begin{itemize}
  \item llm_output \(\leftarrow\) LLM(C.\text{verify} \text{prompt}(s_i, s_{i+1}, C))
  \item if ITP(llm_output) = pass then
    \item return True
  \item else
    \item return False
  \item end if
\end{itemize}

end function
Algorithm 5 Correct

\textbf{Requires:}
- \texttt{LLM} Large Language Model, i.e., GPT-3.5-turbo in our experiments
- \texttt{PARSE} Parse the output of LLM to extract subgoals which correspond to the same segment of the formal sketch as $s_{i+1}$
- \texttt{UPDATE} Replace $s_{i+1}$ in C with newly generated subgoals

\textbf{function} \texttt{CORRECT}(s_i, s_{i+1}, C)
\begin{itemize}
  \item \texttt{llm\_output} $\leftarrow$ \texttt{LLM}(\texttt{C.correct\_prompt}(s_i, s_{i+1}, C))
  \item \texttt{subgoals} $\leftarrow$ \texttt{PARSE}(\texttt{llm\_output}, s_{i+1}, C)
  \item \texttt{C} $\leftarrow$ \texttt{UPDATE}(\texttt{llm\_output}, s_{i+1}, C)
\end{itemize}
\textbf{return} \texttt{subgoals, C}
\textbf{end function}
Your task is to transform a formal proof, written in a style suitable for a theorem-proving system, into a more explanatory and narrative style, suitable for humans. The task involves adding explanatory comments that break down the formal proof into understandable steps, providing context and insight into the underlying logical structure and reasoning.

Input:

```
theorem amc12a_2003_p24:
  fixes a b :: real
  assumes "b < ax < a" and "1 < b
  shows "ln (a/b) / ln a + ln (b/a) / ln b \leq 0" (is "?L \leq _")
proof -
  define x y where "x = ln a" and "y = ln b"
  have "y \geq 0"
    using \langle open \rangle open[0-1] close unfolding y_def using ln_gt_zero sledgehammer
  moreover have "x \geq 0"
    using \langle open \rangle open[0-1] close unfolding x_def y_def using assms(2) sledgehammer
  ultimately have "x \leq y" sledgehammer
  have "L = (x/y) + (y/x)" apply (subt (1 2) ln_div)
    using assms unfolding x_def y_def sledgehammer
  also have "... = x/y + y/x" unfolding power2_eq_square using open[0-1] close open[0-1] close sqt(1) * sqt(y) = 1) close sledgehammer
  finally show ?thesis sledgehammer.
qed.
```

Output:

```
theorem amc12a_2003_p24:
  fixes a b :: real
  assumes "b < ax < a" and "1 < b
  shows "ln (a/b) / ln a + ln (b/a) / ln b \leq 0" (is "?L \leq _")
proof -
  (* Step 1: With the definitions x = ln a, y = ln b, and the constraints y \geq 0 and x \geq y, the equation simplifies into 2 - \frac{y}{x} \leq 0 *)
  define x y where "x = ln a" and "y = ln b"
  have "y \geq 0"
    using \langle open \rangle open[0-1] close unfolding y_def using ln_gt_zero sledgehammer
  moreover have "x \geq 0"
    using \langle open \rangle open[0-1] close unfolding x_def y_def using assms(2) sledgehammer
  ultimately have "x \leq y" sledgehammer
  have "L = (x/y) + (y/x)" apply (subt (1 2) ln_div)
    using assms unfolding x_def y_def sledgehammer
  also have "... = x/y + y/x" unfolding power2_eq_square using open[0-1] close open[0-1] close sqt(1) * sqt(y) = 1) close sledgehammer
  (* Step 2: Using the inequality \(2 - \frac{y}{x} \geq 0\), it is shown that the simplified equation \(2 - \frac{x}{y} \leq 0\) is less than or equal to 0. *)
  also have "... \leq 0" proof -
    have "sqt(x) * sqt(y) = 1" sledgehammer
    have "N < L = (sqt(x) + sqt(y))/2" sledgehammer
    also have "... = x + 2 + y/x" unfolding power2_eq_square using open[0-1] close open[0-1] close sqt(1) * sqt(y) = 1) close sledgehammer
    finally show ?thesis sledgehammer.
qed.
```

Figure 4. Illustration of the Initialize_subgoals function as denoted in Algorithm 1. ChatGPT is leveraged to generate the subgoal-based proof for a formal sketch.

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Your task is to complete the missing part of a Isabelle proof based on the corresponding description provided. The proof will consist of several steps, and you will need to use your knowledge of mathematical concepts and logical reasoning to fill in the gap.

Input:
theorem amc12a_2003_p24:
fixes a b::real
assumes "b ≤ a" and "1<b" shows "ln (a/b) / ln a + ln (b/a) / ln b ≤ 0" (is "?L ≤ 0")
proof –
(* Step 1: With the definitions x = ln a, y = ln b, and the constraints y > 0 and x ≥ y, the equation simplifies into 2 - x - y / x - y / y.

define x y where "x=ln a" and "y=ln b"
have "x > 0" and "y > 0" using assms by (simp_all add: ln_gt_zero)
have "?L = 2 - y / x - x / y" using assms x_def y_def by (simp add: add_divide_distrib diff_divide_distrib ln_div)
(* Step 2: Using the inequality \((x-y)^2 ≥ 0\), it is shown that the simplified equation 2 - x - y / x - y / y is less than or equal to 0. *)
also have "... ≤ 0" proof –
have "sqrt (x/y) * sqrt (y/x) = 1" sledgehammer
have "0 ≤ (sqrt (x/y) - sqrt (y/x))^2" sledgehammer
also have "... = x / y - 2 + y / x" unfolding power2_eq_square using \(x>0\) \(y>0\) \(sqrt (x/y) * sqrt (y/x) = 1\) sledgehammer
finally show ?thesis sledgehammer qed
finally show ?thesis. qed

Output:
theorem amc12a_2003_p24:
fixes a b::real
assumes "b ≤ a" and "1<b" shows "ln (a/b) / ln a + ln (b/a) / ln b ≤ 0" (is "?L ≤ 0")
proof –
(* Step 1: With the definitions x = ln a, y = ln b, and the constraints y > 0 and x ≥ y, the equation simplifies into 2 - x - y / x - y / y.

define x y where "x=ln a" and "y=ln b"
have "x > 0" and "y > 0" using assms by (simp_all add: ln_gt_zero)
have "?L = 2 - y / x - x / y" using assms x_def y_def by (simp add: add_divide_distrib diff_divide_distrib ln_div)
(* Step 2: Using the inequality \((x-y)^2 ≥ 0\), it is shown that the simplified equation 2 - x - y / x - y / y is less than or equal to 0. *)
also have "... ≤ 0" proof –
have "sqrt (x/y) * sqrt (y/x) = 1" sledgehammer
have "0 ≤ (sqrt (x/y) - sqrt (y/x))^2" sledgehammer
also have "... = x / y - 2 + y / x" unfolding power2_eq_square using \(x>0\) \(y>0\) \(sqrt (x/y) * sqrt (y/x) = 1\) sledgehammer
finally show ?thesis sledgehammer qed
finally show ?thesis. qed

Figure 5. An instance of the “verify” component within the Verify_and_correct function in Algorithm 2. ChatGPT encounters a failure in reconstructing the proof associated with step 1, thereby deeming it an unsuitable subgoal.
The Isabelle Proof Assistant was unable to verify Step 1. To resolve this, consider providing more detailed information or clarity to enhance the explanation for this particular step.

Input:

```isar
theorem amc12a_2003_p24:
  fixes a b::real
  assumes "b > 0" and "1 < b" 
  shows "ln (a/b) / ln a + ln (b/a) / ln b \leq 0" (is "?L \leq \_")
proof -
  (* Step 1: With the definitions x = ln a, y = ln b, and the constraints y > 0 and x \geq y, the equation simplifies into 2 - \frac{y}{x} - \frac{x}{y} \leq 0")
  define x y where "x = \ln a" and "y = \ln b" have "x \geq 0"
  using \open[b:0] = close unfolding y_def using ln_gt_zero sledgehammer
  moreover have "x \geq y" using \open[a:ge]\open[b:close] unfolding x_def y_def using assms(2) sledgehammer
  ultimately have "x \geq 0"
  sledgehammer
  have "?L = (x/y) + (y/x)" apply (rule (1 2) ln_div)
  using assms unfolding x_def y_def sledgehammer
  also have "... = x/y + y/x" unfolding power2_eq_square
  sledgehammer
  finally show ?thesis.
qed
```

Output:

```isar
theorem amc12a_2003_p24:
  fixes a b::real
  assumes "b > 0" and "1 < b" 
  shows "ln (a/b) / ln a + ln (b/a) / ln b \leq 0" (is "?L \leq \_")
proof -
  (* Step 1: Define x = ln a and y = ln b. *)
  define x y where "x = ln a" and "y = ln b" (* Step 2: With the conditions set as y \geq 1 and x \geq y, we transition the equation to be formed in terms of x and y, thereby yielding 2 \frac{y}{x} + \frac{x}{y} \leq 0")
  have "x \geq 0"
  using \open[b:0] = close unfolding y_def using ln_gt_zero sledgehammer
  moreover have "x \geq y" using \open[a:ge]\open[b:close] unfolding x_def y_def using assms(2) sledgehammer
  ultimately have "x \geq 0"
  sledgehammer
  have "?L = (x/y) + (y/x)" apply (rule (1 2) ln_div)
  using assms unfolding x_def y_def sledgehammer
  also have "... = x/y + y/x" unfolding power2_eq_square
  sledgehammer
  finally show ?thesis.
qed```

Figure 6. An instance of the “correct” component within the Verify_and_correct function in Algorithm 2. ChatGPT works on the decomposition of the original subgoal (i.e., step 1 in the input) into a series of more granular subgoals (i.e., steps 1 - 3 in the output).
Your task is to complete the missing part of a Isabelle proof based on the corresponding description provided. The proof will consist of several steps, and you will need to use your knowledge of mathematical concepts and logical reasoning to fill in the gap.

**Input:**

```isabelle
theory amc12a_2003_p24:
  fixes a b real
  assumes "0 < x" and "1 < b"  
  shows "ln (a/b) / ln a + ln (b/a) / ln b < 0" (is "?L < 0")

proof -
  (** Step 1: Define x = ln a and y = ln b. **)
  have "y > 0"
  using "open-bounded" unfolding y_def using ln_gt_zero sledgehammer
  moreover have "x < 0"
  using "open-a>0" get=b/ln/b' unfolding x_def y_def using assume(2) sledgehammer
  ultimately have "x < 0" and "y > 0" sledgehammer

  (** Step 2: With the conditions set as y > 0 and x ≥ y, we transition the equation to be framed in terms of x and y, thereby yielding  1 < b. **) 
  have "0 < x" and "y > 0" unfolding x_def y_def sledgehammer
  using assms unfolding x_def y_def sledgehammer
  (** Step 3: Simplify the expression to 2 - x/y is less than or equal to 0. **) 
  have "2 - x/y + x/y" unfolding x_def y_def using sledgehammer
  (** Step 4: Using the inequality ( 2 - x/y + x/y ) ≥ 0, it is shown that the simplified equation 2 - x/y is less or equal to 0. **) 
  have "... = x/y ≤ 2 - x/y" unfolding power2_eq_square using (open-x<0/close) (open-y<0/close) (open-sqrt (x/y) * sqrt (y/x) = 1) sledgehammer
  finally show ?thesis sledgehammer
  qed
```

**Output:**

```isabelle
theory amc12a_2003_p24:
  fixes a b real
  assumes "0 < x" and "1 < b"  
  shows "ln (a/b) / ln a + ln (b/a) / ln b < 0" (is "?L < 0")

proof -
  (** Step 1: Define x = ln a and y = ln b. **)
  define x where "x = ln a" and "y = ln b".
  (** Step 2: With the conditions set as y > 0 and x ≥ y, we transition the equation to be framed in terms of x and y, thereby yielding  1 < b. **) 
  have "y > 0"
  using "open-bounded" unfolding y_def using ln_gt_zero sledgehammer
  moreover have "x < 0"
  using "open-a>0" get=b/ln/b' unfolding x_def y_def using assume(2) sledgehammer
  ultimately have "x < 0" and "y > 0" sledgehammer

  (** Step 2: With the conditions set as y > 0 and x ≥ y, we transition the equation to be framed in terms of x and y, thereby yielding  1 < b. **) 
  have "0 < x" and "y > 0" unfolding x_def y_def sledgehammer
  using assms unfolding x_def y_def sledgehammer
  (** Step 3: Simplify the expression to 2 - x/y is less than or equal to 0. **) 
  have "2 - x/y + x/y" unfolding x_def y_def using sledgehammer
  (** Step 4: Using the inequality ( 2 - x/y + x/y ) ≥ 0, it is shown that the simplified equation 2 - x/y is less or equal to 0. **) 
  have "... = x/y ≤ 2 - x/y" unfolding power2_eq_square using (open-x<0/close) (open-y<0/close) (open-sqrt (x/y) * sqrt (y/x) = 1) sledgehammer
  finally show ?thesis sledgehammer
  qed
```

**Figure 7.** An instance of the “verify” component within the Verify_and_correct function in Algorithm 2. ChatGPT successfully reconstructs the proof associated with step 1, thus validating it as a viable subgoal.

---

18
Your task is to complete the missing part of a Isabelle proof based on the corresponding description provided. The proof will consist of several steps, and you will need to use your knowledge of mathematical concepts and logical reasoning to fill in the gap.

Input:
```isabelle
theory amc12a_2003_p24
  fixes a b :: real
  assumes "b < 0" and "1 < b" shows "ln (a/b) / ln a + ln (b/a) / ln b ≥ 0" (is "?L ≥ 0")
proof -
  (* Step 1: Define s = ln a and y = ln b.
  *)
  define x y where "s = ln a" and "y = ln b."
  (* Step 2: With the conditions set as y > 0 and x ≥ y, we transition the equation to be framed in terms of x and y, thereby yielding
  "x ≥ 0" ∨ "y ≤ 0". *)
  have "x ≥ 0" using open_sum_closure [open: y > 0] close: sledgehammer
  (* Step 3: Simplify the expression to 2 - 2 * (y / x - 2 * y / x). *)
  have "x ≥ 0" by (simp add: ln_gt_zero)
  moreover have "x ≥ 0" using open_sum_closure [open: y > 0] close: sledgehammer
  hence "x - y * y + (y - x) / y = 2 - y / x - x" using open_sum_closure [open: y > 0] close: by (simp add: diff_divide_distrib)
  (* Step 3: Simplify the expression to 2 - 2 * (y / x - 2). *)
  have "x ≥ 0" using open_sum_closure [open: y > 0] close: sledgehammer
  (* Step 4: Using the inequality (sqrt (x/y) * sqrt (y/x))^2 ≥ 0, it is shown that the simplified equation 2 - 2 * (y / x) ≤ 0 or equal to 0. *)
  also have "x ≥ 0" using open_sum_closure [open: y > 0] close: sledgehammer
  using open_sum_closure [open: y > 0] close: by (simp add: diff_divide_distrib)
  have "sqrt (x/y) * sqrt (y/x) = 1" sledgehammer
  finally show ?thesis sledgehammer
qed
```

Output:
```isabelle
theory amc12a_2003_p24
  fixes a b :: real
  assumes "b < 0" and "1 < b" shows "ln (a/b) / ln a + ln (b/a) / ln b ≥ 0" (is "?L ≥ 0")
proof -
  (* Step 1: Define s = ln a and y = ln b.
  *)
  define x y where "s = ln a" and "y = ln b."
  (* Step 2: With the conditions set as y > 0 and x ≥ y, we transition the equation to be framed in terms of x and y, thereby yielding
  "x ≥ 0" ∨ "y ≤ 0". *)
  have "y > 0" using open_sum_closure [open: y > 0] close: unfolding y_def using ln_zero_sledgehammer
  moreover have "x ≥ 0" using open_sum_closure [open: y > 0] close: unfolding x_def using ln_mono_sledgehammer
  hence "(x - y * y + (y - x) / y = 2 - y / x - x)" using open_sum_closure [open: y > 0] close: by (simp add: diff_divide_distrib)
  (* Step 3: Simplify the expression to 2 - 2 * (y / x - 2). *)
  have "x ≥ 0" using open_sum_closure [open: y > 0] close: sledgehammer
  (* Step 4: Using the inequality (sqrt (x/y) * sqrt (y/x))^2 ≥ 0, it is shown that the simplified equation 2 - 2 * (y / x) ≤ 0 or equal to 0. *)
  also have "x ≥ 0" using open_sum_closure [open: y > 0] close: sledgehammer
  using open_sum_closure [open: y > 0] close: by (simp add: diff_divide_distrib)
  have "sqrt (x/y) * sqrt (y/x) = 1" sledgehammer
  finally show ?thesis sledgehammer
qed
```

Figure 8. An instance of the “verify” component within the Verify and Correct function in Algorithm 2. ChatGPT encounters a failure in reconstructing the proof associated with step 2, thereby deeming it an unsuitable subgoal.
Your task is to complete the missing part of a Isabelle proof based on the corresponding description provided. The proof will consist of several steps, and you will need to use your knowledge of mathematical concepts and logical reasoning to fill in the gap.

Input:
theorem amc12a_2003_p24:
fixes a b::real
assumes "0<y" and "y>0" shows "ln (a/b) / ln a + ln (b/a) / ln b < 0" (is "?L < 0")
proof -
(* Step 1: Define x = ln a and y = ln b. *)
define x y where "x = ln a" and "y = ln b"

(* Step 2: With the conditions set as y > 0 and x ≥ y, we transition the equation to be framed in terms of x and y, thereby yielding...

have "y = 0" using (a>b) l<1 unfolding y_def using ln_gt_zero sledgehammer
moreover have "x = y" using (a>b) ge=0 b l<1 unfolding x_def y_def using assms(2) sledgehammer
ultimately have "x = 0"

sledgehammer
have "?L = (x/y) + (y/x)" apply subst (1 2 ln_div)
using assms unfolding x_def y_def sledgehammer

(* Step 3: Simplify the expression to 2 - x / y. *)

(*)

also have "... "≤ 0"

proof -

have "sqrt (x/y) * sqrt (y/x) = 1" sledgehammer
have "2 - (x/y) / y" sledgehammer
also have "... = x/y + x/y" unfolding power2_module_square using open=0 l<0 close
finally show ?thesis sledgehammer

qed.


(* Step 4: Using the inequality (2 - x / y) ≤ 0, it is shown that the simplified equation 2 - x / y is less than or equal to 0. *)

also have "... "≤ 0"

proof -

have "sqrt (x/y) * sqrt (y/x) = 1" sledgehammer
have "2 - (x/y) / y" sledgehammer
also have "... = x/y + x/y" unfolding power2_module_square using open=0 l<0 close
finally show ?thesis sledgehammer

qed.

(* Step 4: Using the inequality (2 - x / y) ≥ 0, it is shown that the simplified equation 2 - x / y is less than or equal to 0. *)

also have "... "≤ 0"

proof -

have "sqrt (x/y) * sqrt (y/x) = 1" sledgehammer
have "2 - (x/y) / y" sledgehammer
also have "... = x/y + x/y" unfolding power2_module_square using open=0 l<0 close
finally show ?thesis sledgehammer

qed.


Output:

theorem amc12a_2003_p24:
fixes a b::real
assumes "0<y" and "y>0" shows "ln (a/b) / ln a + ln (b/a) / ln b < 0" (is "?L < 0")
proof -
(* Step 1: Define x = ln a and y = ln b. *)
define x y where "x = ln a" and "y = ln b"

(* Step 2: With the conditions set as y > 0 and x ≥ y, we transition the equation to be framed in terms of x and y, thereby yielding...

have "y = 0" using (a>b) l<1 unfolding y_def using ln_gt_zero sledgehammer
moreover have "x = y" using (a>b) ge=0 b l<1 unfolding x_def y_def using assms(2) sledgehammer
ultimately have "x = 0"

sledgehammer
have "?L = (x/y) + (y/x)" apply subst (1 2 ln_div)
using assms unfolding x_def y_def sledgehammer

(* Step 3: Simplify the expression to 2 - x / y. *)

(*)

also have "... "≤ 0"

proof -

have "sqrt (x/y) * sqrt (y/x) = 1" sledgehammer
have "2 - (x/y) / y" sledgehammer
also have "... = x/y + x/y" unfolding power2_module_square using open=0 l<0 close
finally show ?thesis sledgehammer

qed.

(* Step 4: Using the inequality (2 - x / y) ≤ 0, it is shown that the simplified equation 2 - x / y is less than or equal to 0. *)

also have "... "≤ 0"

proof -

have "sqrt (x/y) * sqrt (y/x) = 1" sledgehammer
have "2 - (x/y) / y" sledgehammer
also have "... = x/y + x/y" unfolding power2_module_square using open=0 l<0 close
finally show ?thesis sledgehammer

qed.

(* Step 4: Using the inequality (2 - x / y) ≥ 0, it is shown that the simplified equation 2 - x / y is less than or equal to 0. *)

also have "... "≤ 0"

proof -

have "sqrt (x/y) * sqrt (y/x) = 1" sledgehammer
have "2 - (x/y) / y" sledgehammer
also have "... = x/y + x/y" unfolding power2_module_square using open=0 l<0 close
finally show ?thesis sledgehammer

qed.

Figure 9. An instance of the “verify” component within the Verify_and_correct function in Algorithm 2. ChatGPT successfully reconstructs the proof associated with step 3, thus validating it as a viable subgoal.
The Isabelle Proof Assistant was unable to verify Step 2. To resolve this, consider providing more detailed information or clarity to enhance the explanation for this particular step.

Input:

```isar
theorem amc12a_2003_p24:
fixes a b real
assumes \( 1 < b \) and \( 1 < a \)
shows \( \ln (a/b) + \ln (b/a) + \ln b \leq \ln b \) (isabelle):\( 1 < b \) and \( 1 < a \)
proof -(\* Step 1: Define \( s = \ln a \) and \( y = \ln b \) *)
define \( x y \) where \( "s=\ln x " \) and \( "y=\ln y " \)
(* Step 2: Show that \( s \geq y > 0 \) and \( x \geq y \); we transition the equation to be framed in terms of \( x \) and \( y \), thereby yielding \( \frac{\ln x}{\ln y} + \frac{\ln y}{\ln x} + \frac{\ln y}{\ln y} \leq \frac{\ln y}{\ln y} \) *)
also have \( y > 0 \)
proof-(\* Step 3: Use the definitions of \( x \) and \( y \) to rewrite the expression as \( \frac{x}{y} + \frac{y}{x} + \frac{y}{y} \) *)
also have \( \frac{x}{y} \geq y > 0 \)
proof-(\* Step 4: Using the inequality \( \frac{x}{y} + \frac{y}{x} \leq \frac{y}{y} \) *)
also have \( \frac{x}{y} \geq y > 0 \)
proof-(\* Step 5: With the conditions set as \( x \geq y \geq 0 \), we transition the equation to be framed in terms of \( x \) and \( y \), thereby yielding \( \frac{\ln x}{\ln y} + \frac{\ln y}{\ln x} + \frac{\ln y}{\ln y} \leq \frac{\ln y}{\ln y} \) *)
finally show \( \text{thesis} \)
```
Your task is to complete the missing part of an Isabelle proof based on the corresponding description provided. The proof will consist of several steps, and you will need to use your knowledge of mathematical concepts and logical reasoning to fill in the gap.

Input:

```isabelle
proof (isabelle amc12a_2003_p24):
fixes a b :: real
assumes b >0 and "t>0"
shows "ln(ax)/ln x + ln(bx)/ln b =c= 0" (is "")
proof -
(* Step 1: Define x = ln a and y = ln b. *)
define x y where "x=ln a" and "y=ln b"
(* Step 2: Show that y > 0 and x ≥ y. *)

(* Step 3: Use the definitions of x and y to rewrite the expression as \( \frac{x}{y} + \frac{y}{x} \). *)

have "ln(ax)/ln x = (y-x)" by (rule ln_gt_zero)
apply (sub(1 2) ln, div)
using assms unfolding x_def y_def sledgehammer
(* Step 4: Simplify the expression to \( 2 - \frac{x}{y} + \frac{y}{x} \). *)
also have "... = 2 - (y/x + x/y)" using (rule sledgehammer)

(* Step 5: Using the inequality \( (\frac{x}{y} + \frac{y}{x})^2 \geq 0 \), it is shown that the simplified equation 2 - \( \frac{x}{y} + \frac{y}{x} \) is less than or equal to 0. *)
also have "... ≤ 0" using (rule sledgehammer)

-end-
```

Output:

```isabelle
proof (isabelle amc12a_2003_p24):
fixes a b :: real
assumes b >0 and "t>0"
shows "ln(ax)/ln x + ln(bx)/ln b =c= 0" (is "")
proof -
(* Step 1: Define x = ln a and y = ln b. *)
define x y where "x=ln a" and "y=ln b"
(* Step 2: Show that y > 0 and x ≥ y. *)

(* Step 3: Use the definitions of x and y to rewrite the expression as \( \frac{x}{y} + \frac{y}{x} \). *)

have "ln(ax)/ln x = (y-x)" by (rule ln_gt_zero)
apply (sub(1 2) ln, div)
using assms unfolding x_def y_def sledgehammer
moreover have "x ≥ 0" using (rule sledgehammer)
ultimately have "x > 0" using (rule sledgehammer)

(* Step 3: Use the definitions of x and y to rewrite the expression as \( \frac{x}{y} + \frac{y}{x} \). *)

have "ln(ax)/ln x = (y-x)" by (rule ln_gt_zero)
apply (sub(1 2) ln, div)
using assms unfolding x_def y_def sledgehammer
(* Step 4: Simplify the expression to \( 2 - \frac{x}{y} + \frac{y}{x} \). *)
also have "... = 2 - (y/x + x/y)" using (rule sledgehammer)

(* Step 5: Using the inequality \( (\frac{x}{y} + \frac{y}{x})^2 \geq 0 \), it is shown that the simplified equation 2 - \( \frac{x}{y} + \frac{y}{x} \) is less than or equal to 0. *)
also have "... ≤ 0" using (rule sledgehammer)

-end-
```

---

Figure 11. An instance of the “verify” component within the Verify_and_correct function in Algorithm 2. ChatGPT successfully reconstructs the proof associated with step 2, thus validating it as a viable subgoal.
Figure 12. An instance of the “verify” component within the Verify_and_correct function in Algorithm 2. ChatGPT successfully reconstructs the proof associated with step 3, thus validating it as a viable subgoal.
B. More Details about Demonstration Reorganization

B.1. Parameterization

In alignment with Austin et al. (2021), we adopt discrete diffusion models to model binary random variables. Explicitly, the forward process is given by:

\[
q(\psi | \psi_{t-1}) = \text{Cat} (\psi; p = \delta(\psi_{t-1})\mathbf{Q}_t),
\]

where \(\delta(\psi)\) symbolizes the one-hot encoding of \(\psi\), \(\mathbf{Q}_t = \begin{bmatrix} (1 - \beta_t) & \beta_t \\ \beta_t & (1 - \beta_t) \end{bmatrix} \) denotes the transition matrix, \(\beta_t\) corresponds to the corruption ratio and satisfies that \(\prod_{t=1}^{T} (1 - \beta_t) \approx 0\). The marginal at step \(t\) and the posterior at step \(t - 1\) can be articulated as:

\[
q(\psi_t | \psi_0) = \text{Cat} (\psi_t; p = \delta(\psi_0)\overline{\mathbf{Q}}_t),
\]

\[
q(\psi_{t-1} | \psi_t, \psi_0) = \text{Cat} \left( \psi_{t-1}; p = \frac{\delta(\psi_t)\mathbf{Q}_t^\top \otimes \delta(\psi_0)\overline{\mathbf{Q}}_{t-1}}{\delta(\psi_0)\overline{\mathbf{Q}}_t \delta(\psi_t)} \right),
\]

where \(\overline{\mathbf{Q}}_t = \mathbf{Q}_1\mathbf{Q}_2 \ldots \mathbf{Q}_t\). In consonance with Austin et al. (2021), we employ a denoising neural network which is tasked with the prediction of \(p(\psi_0 | \psi_t)\), thereby enabling the parameterization of the reverse process:

\[
p_\theta(\psi_{t-1} | \psi_t, x) \propto \sum_\psi q(\psi_{t-1} | \psi_t, \psi_0) p_\theta(\psi_0 | \psi_t, x).
\]

B.2. Implementation of GNN

Our work employs a modified version of GNN, a model that exhibits anisotropic characteristics and is enhanced by edge gating methodologies (Bresson & Laurent, 2018; Sun & Yang, 2023). We define \(t\) as sinusoidal representations (Vaswani et al., 2017) associated with the denoising timestep \(t\). Consider \(h^\ell_i\) and \(e^\ell_{ij}\) as the features of node \(i\) and edge \(ij\) at a specific layer \(\ell\), respectively. During the transition between layers, these features disseminate via an anisotropic message propagation paradigm as follows:

\[
\begin{align*}
\hat{e}^\ell_{ij} &= P^\ell e^\ell_{ij} + Q^\ell h^\ell_i + R^\ell h^\ell_j, \\
e^{\ell+1}_{ij} &= e^{\ell+1}_{ij} + \text{MLP}_e(\text{BN}(\hat{e}^{\ell+1}_{ij})) + \text{MLP}_e(t), \\
h^{\ell+1}_i &= h^{\ell+1}_i + \text{ReLU}(\text{BN}(U^\ell h^\ell_i + \sum_{j \in \mathcal{N}_i} (\sigma(e^{\ell+1}_{ij}) \otimes V^\ell h^\ell_j))),
\end{align*}
\]

where \(P^\ell, Q^\ell, R^\ell, U^\ell, V^\ell \in \mathbb{R}^{d \times d}\) denote layer-specific learnable parameters with \(d\) denoting the dimension of hidden state. BN signifies the Batch Normalization operation (Ioffe & Szegedy, 2015), while SUM represents sum pooling. \(\otimes\) designates the Hadamard product, and \(\mathcal{N}_i\) encapsulates the set of neighboring nodes of node \(i\). Lastly, a two-layer multi-layer perceptron is denoted by MLP(.)

In our experiments, we define \(h^0_i = W[\text{Ada}(x); \text{Ada}(E^{(K)}_i)]\) where \(W \in \mathbb{R}^{d \times 3072}\) is a learnable parameter. \(\text{Ada}(x), \text{Ada}(E^{(K)}_i) \in \mathbb{R}^{536 \times 1}\) denote the ada embeddings \(^5\) of the statement \(x\) and the \(i\)-th demonstration example, respectively. The operator \([\; ; \;]\) denotes the concatenation operation between two vectors. \(e^0_{ij}\) are initialized as sinusoidal features of the edges.

B.3. Sampling Process

A straightforward strategy for creating a demonstration organization is by directly sampling \(\psi \sim p_\theta(\psi_0 | x)\). However, this strategy introduces two key challenges: (1) A cycle in \(\psi\) might be present, indicating that at least one demonstration example is selected multiple times; (2) \(\psi\) could include multiple separate sub-graphs, making it difficult to define the relative position between two demonstration examples from two different sub-graphs. Taking a cue from treating diffusion models as discriminative approaches (Li et al., 2023), we start by randomly creating 200 potential solutions. Using the diffusion model’s ability to provide conditional density estimates, we rate these 200 potential solutions and select the one with the highest score to build the final demonstration organization. We then reconstruct the sequence of demonstration examples from \(\psi\), adding examples one by one into the LLM context until we hit the length limit of the LLM.

\(^5\)https://platform.openai.com/docs/guides/embeddings
B.4. Hyperparameters and Hardware Setup

In the course of our experiment, we employ a 3-layer Anisotropic Graph Neural Network with a hidden state dimensionality set to 256. We sweep the learning rate from $[1e^{-4}, 2e^{-4}, 5e^{-4}, 7e^{-4}]$ and sweep batch size from $[4, 8, 16, 32]$. The processes of training and inference for the diffusion models are conducted on an NVIDIA RTX 3090 GPU.

C. More In-depth Analysis

C.1. Impact of Prompt Wording

To systematically investigate the influence of varying prompt wordings on the effectiveness of our proposed method, we further conduct an experiment on the miniF2F-test set. The specific prompt is as follows:

Assume the role of a mathematician proficient in Isabelle.
When provided with both informal and formal statements of a problem,
your responsibility is to formulate a formal proof that Isabelle can verify.

Table 3. Comparison of pass rates with different prompt wordings over 100 autoformalization attempts on the miniF2F-test set.

<table>
<thead>
<tr>
<th>Prompt Type</th>
<th>pass@100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original (Proposed)</td>
<td>45.5%</td>
</tr>
<tr>
<td>Alternative Prompt</td>
<td>41.0%</td>
</tr>
</tbody>
</table>

As can be observed in Table 3, the original prompt yielded slightly better results compared to the alternative prompt. This suggests that the phrasing of the prompt can have an influence on the performance, and our original choice was more effective in achieving higher pass rates.

C.2. Sensitivity to Example Ordering

To assess the robustness and adaptability of our model, we explored the sensitivity of the model to the ordering of examples. Specifically, we maintained the examples derived from our diffusion model but altered their sequence. The experiment involves randomizing the sequence of examples after they were organized using the diffusion model. We then evaluated the model’s ability to solve problems under this modified setup. The evaluation was conducted over 20 autoformalization attempts on the miniF2F-test set.

Table 4. Impact of example ordering.

<table>
<thead>
<tr>
<th>Method</th>
<th># of Problems Solved</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ours (Ordered)</td>
<td>94</td>
</tr>
<tr>
<td>Shuffled Order</td>
<td>39</td>
</tr>
</tbody>
</table>

The results in Table 4 demonstrate that the ordering of examples significantly impacts the model’s performance. It is evident that the structured order provided by the diffusion model is important, as disrupting this order resulted in a substantial decline in the number of problems solved.

C.3. Impact of Demonstration Selection and Presentation Order

We conducted experiments comparing our method, which meticulously selects and orders demonstrations, to an alternative approach where the demonstrations are shuffled.

The results in Table 5 underscore the significance of both the selection and the order in which demonstrations are presented. It is evident that a methodical approach to selecting and ordering demonstrations yields superior results compared to a scenario where demonstrations are presented in a shuffled manner.


Table 5. Comparison of our method vs. shuffled demonstrations

<table>
<thead>
<tr>
<th>Method</th>
<th># of Problems Solved</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ours</td>
<td>94</td>
</tr>
<tr>
<td>Shuffled</td>
<td>39</td>
</tr>
</tbody>
</table>

C.4. Impact of Increasing Demonstrations

We conduct experiments to elucidate the impact of varying the number of demonstrations on the performance of our model.

Table 6. Performance analysis with varying numbers of demonstrations.

<table>
<thead>
<tr>
<th>Number of Demonstrations</th>
<th># of Problems Solved</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ours</td>
<td>94</td>
</tr>
<tr>
<td>1 example</td>
<td>16</td>
</tr>
<tr>
<td>2 examples</td>
<td>29</td>
</tr>
<tr>
<td>3 examples</td>
<td>34</td>
</tr>
</tbody>
</table>

The empirical results in Table 6 underscore the collective significance of all demonstrations in enhancing the model’s performance. It is evident that the incorporation of multiple demonstrations is pivotal, substantiating the efficacy of our diffusion-based model.

C.5. Impact of Randomly Generated Problem Names

The incorporation of original problem names could potentially facilitate models like ChatGPT in associating them to readily available solutions online. This scenario posits a risk of primarily assessing the model’s capability to translate human proof sketches to subgoals and formal proofs, rather than its proficiency in innovating novel subgoals. To mitigate this concern and uphold the rigor of our evaluation, we initiated experiments substituting original problem names with random identifiers and analyzed the ensuing impact on performance. The evaluation was conducted over 20 autoformalization attempts on the miniF2F-test set.

Table 7. Performance comparison with original and random problem names on the miniF2F-test set.

<table>
<thead>
<tr>
<th>Method</th>
<th>Original</th>
<th>Random</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ours</td>
<td>38.5%</td>
<td>39.8%</td>
</tr>
<tr>
<td>Top-K</td>
<td>35.7%</td>
<td>34.8%</td>
</tr>
</tbody>
</table>

The results in Table 7 indicate that our method maintains robust performance, even manifesting a marginal enhancement when random names are deployed. This steadfastness in performance, regardless of the naming conventions adopted, accentuates the robustness and adaptability of our approach across diverse experimental conditions.

C.6. Impact of Proof Sketch Quality

To comprehensively assess the robustness of our method in relation to the quality of informal proof sketches, we conducted an additional experiment that involved the generation of informal proofs using gpt-3.5-turbo-0613 for problems from the miniF2F-valid dataset. Utilizing the methodology outlined in §2.1, we constructed subgoal-based proofs for these informal proofs. For this purpose, a set of 61 problems was carefully chosen to serve as demonstration samples, aligning with the experimental setup specified in §3.4. The results of this experiment are shown in Table 8.

The results from Table 8 demonstrate that our method maintains a notable degree of robustness against variations in the quality of informal proof sketches. This robustness indicates the adaptability and dependability of our approach when confronted with different levels of proof sketch quality, affirming its practical applicability in diverse scenarios.
Table 8. Performance comparison with varying proof sketch quality.

<table>
<thead>
<tr>
<th>Method</th>
<th>valid % (±)</th>
<th>test % (±)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ours (DSP Informal Proofs)</td>
<td>48.0% (0.4)</td>
<td>45.5% (0.6)</td>
</tr>
<tr>
<td>Ours (GPT-3.5 Informal Proofs)</td>
<td>47.7% (0.5)</td>
<td>45.0% (0.7)</td>
</tr>
</tbody>
</table>

C.7. Evaluation of Optimal Organization Search

To further evaluate the efficiency of our method, we conducted experiments involving 200 randomized organization sampling attempts for each statement within the miniF2F-valid and miniF2F-test datasets. These efforts aimed to determine the potential of finding an optimal organization for each statement. The results of this extensive search are displayed in Table 9.

Table 9. Performance comparison with optimal organization search.

<table>
<thead>
<tr>
<th>Method</th>
<th>valid %</th>
<th>test %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ours</td>
<td>48.0%</td>
<td>45.5%</td>
</tr>
<tr>
<td>Optimal Organization Search</td>
<td>48.4%</td>
<td>44.3%</td>
</tr>
</tbody>
</table>

These results corroborate previous observations (Jiang et al., 2023) that the advantages of extensive organization searches tend to level off beyond a certain number of attempts (in this case, 100). These results highlight the inherent challenges of the miniF2F dataset and emphasize the effectiveness of our diffusion model.

C.8. Analysis of Sledgehammer Usage

To provide a comprehensive understanding of how our method and DSP (Jiang et al., 2023) utilize the Sledgehammer tool, we have conducted a detailed analysis. This analysis was carried out on a machine with 64 CPU cores, focusing on the average number of Sledgehammer calls and their execution times for each solved statement. The results are shown in Table 10, which illustrates the average number of Sledgehammer calls and their corresponding durations (in seconds) for each method.

Table 10. Average number of Sledgehammer calls and durations.

<table>
<thead>
<tr>
<th>Method</th>
<th>Calls valid</th>
<th>Calls test</th>
<th>Duration (seconds) valid</th>
<th>Duration (seconds) test</th>
</tr>
</thead>
<tbody>
<tr>
<td>DSP</td>
<td>2.33</td>
<td>3.29</td>
<td>2.98</td>
<td>4.94</td>
</tr>
<tr>
<td>Ours</td>
<td>2.88</td>
<td>4.16</td>
<td>4.94</td>
<td>4.94</td>
</tr>
</tbody>
</table>

The results in Table 10 indicate that our method exhibits a slight increase in both the frequency of Sledgehammer calls and their execution times in comparison with DSP. Specifically, this increase is primarily observed in statements that our method can solve but DSP cannot. For these statements, the number of Sledgehammer calls on the miniF2F-valid and miniF2F-test are 3.21 and 5.38, respectively. This suggests that the need for Sledgehammer becomes more important as the problem’s complexity increases.

D. Discussions about Correlation Between Input Statements and Organization

We further investigate the relationship between input statements and the demonstration organization generated by diffusion models within the miniF2F dataset. We aimed to determine whether the diffusion model tends to generate a generic organization broadly applicable across various statements or if it tailors unique organizations to individual statements.

The empirical results suggest that there is not a “one-size-fits-all” optimal demonstration organization. Specifically, even the most adaptable organization we identified could successfully prove only 3 distinct statements. Furthermore, we only observed 4 instances of such adaptable organizations on the miniF2F-test. This highlights the tailored nature of each organization to its corresponding statement. The limited scope of even the most adaptable organization indicates the challenge in identifying the
optimal demonstration organization for each unique statement. This complexity further emphasizes the effectiveness of our model in efficiently searching for and determining the most suitable demonstration organization for individual statements.

E. Discussions about the Train/Test Data Leakage

The potential for train/test leakage, stemming from the use of original problem names from the miniF2F dataset, was brought to our attention. In response, we undertook a thorough analysis. Our findings indicated that about 32.0% of problems could be identifiable by ChatGPT based on their names, and approximately 40% of informal statements were susceptible to leakage. To further elucidate this, a comprehensive manual search was conducted on the test set’s informal statements and their corresponding proofs. This search revealed that around 40% had been available online before September 1, 2021, with platforms like Mathway and Mathematics Stack Exchange being significant contributors. Despite several miniF2F problems lacking Isabelle solutions, we revised the potentially leaked statements. Experiments with randomized identifiers further confirmed that our method’s performance is not contingent on data leakage (see Appendix C.5).

F. Additional Examples

We provide additional cases in Figure 13 to 18 to demonstrate the efficacy of our method.6

Figures 13 to 15 display examples where our method effectively guides the proof process. For instance, in Figure 13, the method leverages demonstrations involving division and modulus operations, successfully navigating towards a clear proof path. Similarly, in Figure 14, it utilizes demonstrations based on squaring and square root operations, accurately predicting potential subgoals. Figure 15 continues this trend, demonstrating our method’s consistent ability to discern viable subgoals, thus facilitating the construction of proofs.

Additionally, Figures 16 to 18 display the same statements as Figures 13 to 15 but with the demonstration examples reordered. This reorganization illustrates the impact of demonstration order on the generation of proofs, revealing how changes in the order can lead to unexpected errors. The comparison between these two groups highlights the significance of demonstration organization in influencing final performance.

---

6In Figures 13 to 18, identical statements are guided by a consistent set of demonstration examples, but in a different order. Some less relevant demonstration examples have been omitted due to space constraints.
As a mathematician familiar with Isabelle, your task is to provide a formal proof in response to a given problem statement. Your proof should be structured and clearly written, meeting the following criteria:

- It can be verified by Isabelle.
- Each step of the proof should be explained in detail using comments enclosed in "(*" and ")".
- The explanation for each step should be clear and concise, avoiding any unnecessary or apostrophistic language.

Please ensure that your proof is well-organized and easy to follow, with each step building upon the previous one.

**Problem:**
Let n be an integer and p be a prime number. Show that p divides n if and only if n is congruent to 0 modulo p.

**Proof:**

```
(* Step 1: Prove the base case n = 0. *)

assume "n = 0" unfolding inmultin1m2n

moreover have "n mod p = 0" using sledgehammer

next assume "n mod p = 0" case n = 0

finally have "n mod p = 0" using sledgehammer

qed
```

**Problem:**
Prove that for any natural number n, 3 divides \( n^3 + 2n + 3n^2 + 3n + 3 \).

**Proof:**

```
(* Step 1: Assume that \( n^3 + 2n + 3n^2 + 3n + 3 \) is congruent to 0 modulo 3. *)

assume "(n^3+2*n+3*n^2+3*n+3) mod 3 = 0" unfolding inmultin1m2n

next assume "(n^3+2*n+3*n^2+3*n+3) mod 3 = 0" case n = 0

finally have "(n^3+2*n+3*n^2+3*n+3) mod 3 = 0" using sledgehammer

qed
```

**Problem:**
Suppose \( n \) is a natural number such that \( n \) leaves a remainder of 3 when divided by 7. Show that \( (2n+1) \) leaves a remainder of 0 when divided by 7.

**Proof:**

```
(* Step 1: Express \( n \) as \( 7k + 3 \) for some \( k \) in \( \mathbb{N} \). *)

assume "n mod 7 = 3 mod 7" unfolding inmultin1m2n

next assume "n mod 7 = 3 mod 7" case n = 0

finally have "n = 7k + 3" using sledgehammer

qed
```

Figure 13. A case from miniF2F-valid. Our method identifies viable subgoals and leverages demonstration examples with division and modulus operations to complete the proof.
Suppose that \( x \) is a positive real number. Show that \( 2 - \sqrt{2} x \geq 2 - x - \frac{1}{x^2} \).

Problem: Suppose that \( x \) and \( y \) are two real numbers such that \( x + y = 13 \) and \( xy = 24 \). Find \( \sqrt{x^2 + y^2} \) and show that it is equal to 11.

Proof: Everest mathml algebra_132: fixes x y :: real and h1 : "xy=24" and h2 : "x+y=13" shows "sqrt(x^2+y^2)=11".

\[
\begin{align*}
\text{Step 1:} & \quad \text{Use the identity } x^2 + y^2 = (x+y)^2 - 2xy \text{ to find } x^2 + y^2 \text{ in terms of } x \text{ and } y. \\
\text{Step 2:} & \quad \text{Substitute the given values of } x \text{ and } y \text{ to find } x^2 + y^2. \\
\text{Step 3:} & \quad \text{Show that } x^2 + y^2 = 121. \\
\text{Step 4:} & \quad \text{Take the square root of both sides to find } \sqrt{x^2 + y^2}. \\
\text{Step 5:} & \quad \text{Show that } \sqrt{x^2 + y^2} = 11. \\
\text{qed}
\end{align*}
\]

Problem: Suppose that \( x \) and \( y \) are two real numbers such that \( \frac{1}{x+y} = 7 \) and \( \sqrt{xy} = \sqrt{5} \). Find \( x^2 + y^2 \).

Proof: Everest mathml algebra_150: fixes x y :: real and h1 : "1/(x+y)=7" and h2 : "sqrt(xy)=sqrt(5)" shows "x^2+y^2=115".

\[
\begin{align*}
\text{Step 1:} & \quad \text{Use the identity } (x+y)(x+y) = x^2 + 2xy + y^2 \text{ to find } x^2 + y^2 \text{ in terms of } x \text{ and } y. \\
\text{Step 2:} & \quad \text{Substitute the given values of } x \text{ and } y \text{ to find } x^2 + y^2. \\
\text{Step 3:} & \quad \text{Show that } x^2 + y^2 = 115. \\
\text{qed}
\end{align*}
\]

\[\text{Figure 14. A case from miniF2F-test. Our method predicts viable subgoals and utilizes demonstration examples involving squaring and square root operations to finalize the proof.}\]
Problem:
Given a real number $n$ such that $n = 3$ and $\frac{n+5}{n^2} = 2$, show that $n = 11$.

Proof:
theorem math_algebra_181: fixes n :: real
assumes H: "n > 0" and h1: "(n+5)/(n^2) = 2" shows "n=11" (* Step 1: Use the given equation $\frac{n+5}{n^2} = 2$ to obtain an equation involving n. *)
proof - (* Step 2: Simplify the equation to obtain an expression for n. *)
  have "(n+5)/(n^2) = 2" using H h1 (* Step 2: Use the definition of factorial to simplify the expression. *)
  have "n+5 = 2 * (n+1) * (n+2)" (* Step 1: Use the definition of factorial to simplify the expression. *)
  ultimately have "n = 11" (* Step 3: Combine the two steps to complete the proof. *)
qed

Problem:
Prove by induction that the sum of the first $n$ odd numbers is $n^2$.

Proof:
theorem induction_sum_odd: fixes n :: nat
assumes "n > 0" shows "\sum_{k=1}^{n} (2k-1) = n^2" (* Step 1: Use the definition of factorial to simplify the expression. *)
proof (induct n)
  (* Step 1: Base case: Show that the sum of the first odd number is 1. *)
  case (Suc n)
  have "\sum_{k=1}^{n+1} (2k-1) = 2 * (n+1) + 1" (* Step 2: Simplify the equation to obtain an expression involving n. *)
  ultimately show "\sum_{k=1}^{n+1} (2k-1) = (n+1)^2" (* Step 3: Combine the two steps to complete the proof. *)
qed simp

Problem:
For any natural number $n$ greater than or equal to 9, there exists a natural number $x$ such that $x^2$ is equal to the quotient of the difference between the factorial of $n + 2$ and the factorial of $n + 1$ divided by the factorial of $n$.

Proof:
theorem ame12b_2020_p6: fixes n :: nat
assumes H: "n < \frac{k}{\sqrt{k}}" shows "\frac{\text{fact} \ (n+2) \ - \ \text{fact} \ n \ \text{mod} \ \text{fact} \ n}{\text{fact} \ n} = \frac{\text{fact} \ (n+1) \ \text{mod} \ \text{fact} \ (n+1) \ \text{fact} \ n}{\text{fact} \ n}" (* Step 1: Use the definition of factorial to simplify the expression. *)
proof - (* Step 2: Rearrange the expression to obtain a perfect square. *)
  have "\frac{\text{fact} \ (n+2) - \ \text{fact} \ n \ \text{mod} \ \text{fact} \ n}{\text{fact} \ n} = \frac{\text{fact} \ (n+1) \ \text{mod} \ \text{fact} \ (n+1) \ \text{fact} \ n}{\text{fact} \ n}" (* Step 2: Rearrange the expression to obtain a perfect square. *)
  show "\frac{\text{fact} \ (n+2) - \ \text{fact} \ n \ \text{mod} \ \text{fact} \ n}{\text{fact} \ n} = \frac{\text{fact} \ (n+1) \ \text{mod} \ \text{fact} \ (n+1) \ \text{fact} \ n}{\text{fact} \ n}" (* Step 3: Show that the perfect square is a natural number. *)
  finally show "\text{thesis using assms} sledgehammer" qed

Figure 15. A case from miniF2F-test. Our method recognizes viable subgoals and successfully finishes the proof.
As a mathematician familiar with Isabelle, your task is to provide a formal proof in response to a given problem statement.

Your proof should be structured and clearly written, meeting the following criteria:
- It can be verified by Isabelle.

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- It can be verified by Isabelle.

Problem:
Let \( n \) be an integer and \( p \) a prime number. Show that \( p \) divides \( n \) if and only if \( n^2 \equiv 0 \pmod{p} \).

Proof:

\begin{verbatim}
proof (induction n)
  case 0
    then show ?case by (sledgehammer)
  next assume
    then show ?case by (sledgehammer)
  ultimately have ?case by (sledgehammer)
proof
  (* Step 1: Assume that \( p \) divides \( n \). Show that \( n^2 \equiv 0 \pmod{p} \).)
  assume \( p \) divides \( n \).
  then show \( n \equiv 0 \pmod{p} \) by (sledgehammer)
  (* Step 2: Assume that \( n^2 \equiv 0 \pmod{p} \). Show that \( p \) divides \( n \).)
  assume \( n \equiv 0 \pmod{p} \).
  then show \( p \) divides \( n \) by (sledgehammer)
  ultimately have \( p \) divides \( n \) by (sledgehammer)
end
proof

Problem:
Suppose \( n \) is a natural number such that \( n \) leaves a remainder of 3 when divided by 7. Show that \( (2n + 1) \) leaves a remainder of 1 when divided by 7.

Proof:

\begin{verbatim}
theorem numbertheory_prmdivnsqmodpeq0:
  fixes n :: int
  assumes \( \text{prime } p \) \n  shows \( \text{if and only if } 3 \pmod{7} \) \n  using \( n^3+2*n+3 \pmod{7} \).
proof -
  (* Step 1: Express \( n \) in terms of \( k+3 \) for some \( k \in \mathbb{N} \).)
  have \( n = 3 \pmod{7} \) using \( n^3+2*n+3 \pmod{7} \).
  (* Step 2: Substitute the expression for \( n \) into \( (2n + 1) \) and simplify.
  \( (2n + 1) = 2*3 \pmod{7} \) \n  ultimately have \( (2n + 1) = 6 \pmod{7} \).
  (* Step 3: Show that the result from Step 2 leaves a remainder of 1 when divided by 7.
  have \( (2n + 1) \equiv 1 \pmod{7} \) \n  ultimately have \( (2n + 1) \equiv 1 \pmod{7} \).
end

Figure 16. A case from miniF2F-valid. The shuffled demonstration organization results in the incorrect application of the "mod_mult_self2" theorem.

Subgoal-based Demonstration Learning for Formal Theorem Proving
As a mathematician familiar with Isabelle, your task is to provide a formal proof in response to a given problem statement.

Your proof should be structured and clearly written, meeting the following criteria:
- It can be verified by Isabelle.
- Each step of the proof should be explained in detail using common mathematical language.
- The explanation for each step should be clear and concise, avoiding any unnecessary or pedagogic language.

Please ensure that your proof is well-organized and easy to follow, with each step building upon the previous one.

Problem:
Suppose that \( x \) and \( y \) are two real numbers such that \( x + y = 13 \) and \( xy = 24 \). Find \( \sqrt{x^2 + y^2} \) and show that it is equal to 11.

Proof:
```isabelle
proof
  { assume x: real, y: real
    shows \( "x+y=11" \)
    { (* Step 1: Use the identity \( x^2 + y^2 = (x + y)^2 - 2xy \) to find \( x^2 + y^2 \) in terms of \( x \) and \( y \). *)
      show \( x^2 + y^2 = 121 \).
    }
    { (* Step 2: Substitute the given values of \( x \) and \( y \) to find \( x^2 + y^2 \). *)
      show \( x^2 + y^2 = 121. \)
    }
    { (* Step 3: Factor the expression to get \( x^2 + y^2 \). *)
      show \( x^2 + y^2 = 121. \)
    }
    { (* Step 4: Conclude that the inequality is true. *)
      have \( "x^2 + y^2 = 121" \) using sledgehammer.
    }
    { (* Step 5: Show that \( \sqrt{x^2 + y^2} \) is a positive real number. Show that \( \sqrt{x^2 + y^2} = 11 \). *)
      have \( \sqrt{x^2 + y^2} = 11 \) using sledgehammer.
    }
  qed
```
```isabelle
problem
  { assume x: real, y: real
    shows \( "x+y=11" \)
    { (* Step 0: Use the identity \( x^2 + y^2 = (x + y)^2 - 2xy \) to find \( x^2 + y^2 \) in terms of \( x \) and \( y \). *)
      have \( x^2 + y^2 = 121 \).
    }
    { (* Step 1: Show that \( x^2 + y^2 \) is a positive real number. *)
      have \( \sqrt{x^2 + y^2} = 11 \) using sledgehammer.
    }
    { (* Step 2: Conclude that the inequality is true. *)
      have \( x^2 + y^2 = 121 \) using sledgehammer.
    }
    { (* Step 3: Show that \( \sqrt{x^2 + y^2} \). *)
      have \( \sqrt{x^2 + y^2} = 11 \) using sledgehammer.
    }
    { (* Step 4: Conclude that the inequality is true. *)
      have \( x^2 + y^2 = 121 \) using sledgehammer.
    }
    { (* Step 5: Show that \( \sqrt{x^2 + y^2} \). *)
      have \( \sqrt{x^2 + y^2} = 11 \) using sledgehammer.
    }
  qed
```

Figure 17. A case from miniF2F-test. The shuffled demonstration organization leads to an unsuitable conjecture, “... = 27^2 – 219,” which cannot be proved using the Sledgehammer.
As a mathematician familiar with Isabelle, your task is to provide a formal proof in response to a given problem statement. Your proof should be structured and clearly written, meeting the following criteria:
- It can be verified by Isabelle.
- Each step of the proof should be explained in detail using comments enclosed in "(*" and ")*."
- The explanation for each step should be clear and concise, avoiding any unnecessary or apologetic language.
- Your proof should be structured and clearly written, meeting the following criteria:
- As a mathematician familiar with Isabelle, your task is to provide a formal proof in response to a given problem statement. Your proof should be structured and clearly written, meeting the following criteria:
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- As a mathematician familiar with Isabelle, your task is to provide a formal proof in response to a given problem statement. Your proof should be structured and clearly written, meeting the following criteria:
- It can be verified by Isabelle.
- Each step of the proof should be explained in detail using comments enclosed in "(*" and ")*."
- The explanation for each step should be clear and concise, avoiding any unnecessary or apologetic language.
- Your proof should be structured and clearly written, meeting the following criteria:
- As a mathematician familiar with Isabelle, your task is to provide a formal proof in response to a given problem statement. Your proof should be structured and clearly written, meeting the following criteria:
- It can be verified by Isabelle.
- Each step of the proof should be explained in detail using comments enclosed in "(*" and ")*."
- The explanation for each step should be clear and concise, avoiding any unnecessary or apologetic language.
- Your proof should be structured and clearly written, meeting the following criteria:

**Problem:**
Given a real number $x$ such that $n = 11$ and $x = 2$, show that $n = 2$.

**Proof:**
Using the definition of factorial, we have $n! = n * (n-1) * ... * 2 * 1$.

**Step 1:** Base case: Show that the sum of the first odd number is 1.

**Step 2:** Inductive step: Assume that the sum of the first $n$ odd numbers is $n^2$.

**Step 3:** Combine the two steps to complete the proof. 

Please ensure that your proof is well-organized and easy to follow, with each step building upon the previous one.

---

**Problem:**
Prove by induction that the sum of the first $n$ odd numbers is $n^2$.

**Proof:**

```latex
theorem induction_sum_odd:
  fixes n :: nat
  assumes "n > 0"
  shows "\sum_{k=0}^{n-1} (2k + 1) = n^2"
proof
  (* Step 1: Simplify the expression on the right
   * Step 2: Rearrange the equation to obtain an expression for $n^2$
   * Step 3: Combine the two steps to complete the proof. *)
  using
  have "\sum_{k=0}^{n-1} (2k + 1) = \sum_{k=0}^{n-1} 2k + \sum_{k=0}^{n-1} 1"
  then show "\sum_{k=0}^{n-1} (2k + 1) = \sum_{k=0}^{n-1} 2k + (n-1)"
  using sledgehammer
  qed

  (* Step 1: Base case: Show that the sum of the first odd number is 1. *)
  case (Suc n)
  have "\sum_{k=0}^{n} (2k + 1) = (n+1)^2 - 1" using sledgehammer
  then show "\sum_{k=0}^{n} (2k + 1) = (n+1)^2 - 1" using sledgehammer
  qed

  (* Step 2: Inductive step: Assume that the sum of the first $n$ odd numbers is $n^2$.
   * Step 3: Combine the two steps to complete the proof. *)
  case (Suc n)
  have "\sum_{k=0}^{n} (2k + 1) = (n+1)^2 - 1" using sledgehammer
  then show "\sum_{k=0}^{n} (2k + 1) = (n+1)^2 - 1" using sledgehammer
  qed

  (* Step 3: Combine the two steps to complete the proof. *)
  ultimately show "n = 0 divides Sum "
  qed simp
```

**Problem:**
For any natural number $n$ greater than or equal to 9, there exists a natural number $x$ such that $x^2$ is equal to the quotient of the difference between the factorial of $n + 2$ and the factorial of $n + 1$ divided by the factorial of $n$.

**Proof:**

```latex
theorem ame12b_2020_p6:
  fixes n :: nat
  assumes "n > 9"
  shows "/\{\text{nat_of_nat } n^2\} / (\text{fact } n) = \text{fact } (n + 2) / \text{fact } (n + 1) / \text{fact } n"
proof
  (* Step 1: Simplify the expression on the right-hand side of the equation using the definition of factorial.
   * Step 2: Rearrange the equation to obtain an expression for $x^2$
   * Step 3: Show that the expression for $x^2$ is a perfect square.
   * Step 4: Conclude that there exists a natural number $x$ such that $x^2$ is equal to the given expression. *)
  have "\text{fact } (n + 2) / \text{fact } (n + 1) / \text{fact } n = \text{fact } (n + 2) / \text{fact } (n + 1) / \text{fact } n" by simp add: fact_reduce
  show "n \leq 9" using sledgehammer
  qed
```

**Figure 18:** A case from miniF2F-test. The shuffling of the optimal demonstration organization leads to a complex and unresolved equation, which further results in a failure to properly simplify the factorial expression.